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On Bernstein's Inequality

Let $C_{2\pi}^{\infty}$ be the set of 2π-periodic, infinitely differentiable real functions on R, the set of real numbers, and put $\|f\| = \max\{|f(x)|: x \in R\}$. For $n \geqslant 0$, set

$$S_n = \left\{ p \in C_{2\pi}^{\infty} : p(x) = \sum_{j=0}^{n} (a_j \cos jx + b_j \sin jx) \right\}.$$

the set of trigonometric polynomials of degree n.

The following result due to Bernstein is well known to any student of approximation theory and is an important result in analysis.

THEOREM 1 (Bernstein [1, Vol. I, p. 26]).

$$p \in S_n, \qquad \|p\| \leqslant M \Rightarrow \|p^{(k)}\| \leqslant Mn^k.$$

What is not so well known is that this derivative condition is a strong one; i.e., the trigonometric polynomials are the only functions that satisfy it for infinitely many values of k. For the student in approximation theory, this second result is even of more interest since a simple application of Jackson's theorem will prove it. Thus,

THEOREM 2. *Let $f \in C_{2\pi}^{\infty}$; then, we have:*

$$\left.\begin{array}{r} \exists M, \theta \geqslant 0 \ni \|f^{(k)}\| \leqslant M\theta^k \\ \textit{for infinitely many } k \end{array}\right\} \Rightarrow \exists n \leqslant \theta \ni f \in S_n,$$

Proof. Let $m = 1 + [\theta]$ and consider the best S_m approximation p_0 to f, minimizing $\|f - p\|$. According to Jackson's theorem [2, p. 55], $\|f - p_0\| \leqslant c\|f^{(k)}\|/m^k$ for all k and for some constant c. Letting $k \to \infty$ carefully yields $p_0 = f \in S_m$. Moreover, a_m and b_m of p_0 are zero, as otherwise, $\|f^{(k)}\| \neq O(\theta^k)$. Thus, $f \in S_n$, $n = [\theta]$.

Finally, it should be remarked that (i) a similar proof can be given from estimates of Fourier coefficients given in Zygmund [3, p. 71], and (ii) that limiting the rate of growth of derivatives severly restricts the higher Fourier coefficients is mentioned in books on harmonic analysis.

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